Lovelock black holes with a nonlinear Maxwell field

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We derive electrically charged black hole solutions of the Einstein-Gauss-Bonnet equations with a nonlinear electrodynamics source in $n(\geq 5)$ dimensions. The spacetimes are given as a warped product $\mathcal{M}^2 \times \mathcal{K}^{n-2}$, where $\mathcal{K}^{n-2}$ is a $(n-2)$-dimensional constant curvature space. We establish a generalized Birkhoff's theorem by showing that it is the unique electrically charged solution with this isometry and for which the orbit of the warp factor on $\mathcal{K}^{n-2}$ is non-null. An extension of the analysis for full Lovelock gravity is also achieved with a particular attention to the Chern-Simons case.

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I. INTRODUCTION

Gravitation physics in higher dimensions has been recently investigated in a focused way mainly motivated by string theory. Higher-dimensional general relativity is realized in the lowest order of the Regge slope expansion of strings. Even in general relativity, black holes in higher dimensions have much richer structures than those in four dimensions [1]. The next stringy correction yields the quadratic Riemann curvature terms in the heterotic string case [2, 3]. In order that the graviton amplitude is ghost-free, a special combination of the remaining curvature-squared terms is required yielding to the renormalizable Gauss-Bonnet term [4].

The origin of considering higher-order curvature invariants lies in the attempt of generalizing the theory of general relativity in higher dimensions. Indeed, under the standard assumptions of general relativity it is natural to describe the spacetime geometry in three and four dimensions by the Einstein-Hilbert action while for dimensions greater than four, a more general theory is available. This fact has been first noticed by Lanczos [5] in five dimensions and later generalized by Lovelock [6] for arbitrary dimensions n.

The resulting theory is described by the so-called Lovelock Lagrangian which is a n-form constructed with the vielbein e^a, the spin connection ω^{ab}, and their exterior derivatives without using the Hodge dual. The Lagrangian is a polynomial of degree $[n/2]$ in the curvature two-form, $R^{ab} = d\,\omega^{ab} + \omega^{a}_{\ c} \wedge \omega^{cb}$, given by

$$\mathcal{L}^{(n)} = \sum_{p=0}^{[\frac{n-1}{2}]} \alpha_p \ \epsilon_{a_1 \cdots a_n} R^{a_1 a_2} \cdots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \cdots e^{a_n} \ , \tag{1.1}$$

where $[x]$ denotes the integer part of x, α_p being arbitrary dimensionful coupling constants and wedge products between forms are understood. The corresponding action contains the same degrees of freedom as the Einstein-Hilbert action [7].

The local Lorentz invariance of the Lovelock action (1.1) can be extended into a local (anti-)de Sitter ((A)dS) symmetry in odd dimensions by fixing properly the Lovelock coefficients α_p. For the AdS case the coefficients are given by

$$\alpha_p = \frac{1}{n-2p} \begin{pmatrix} [\frac{n-1}{2}] \\ p \end{pmatrix}, \tag{1.2}$$

where the AdS radius was set equal to 1. The resulting Lagrangian belongs to the class of Chern-Simons gauge theories with Yang-Mills gauge symmetries, and admit supersymmetric extensions. (See [8] and references therein.)

It is clear that these higher-curvature terms come into play in extremely curved regions. Black holes and singularities are one of the best testbeds for demonstrating the effects of these higher-curvature terms. There exists an extensive literature about the exact black-hole solutions, the thermodynamics, the stability, and other topics concerning the Gauss-Bonnet or more generally the Lovelock theory. (See [9, 10] for detailed recent reviews on the subject.)

In the present paper, we shall consider the Gauss-Bonnet and more generally the Lovelock action in presence of a nonlinear electrodynamics source given as an arbitrary power q of the Maxwell invariant,

$$\int d^n x \sqrt{-g} (F_{\mu\nu} F^{\mu\nu})^q. \tag{1.3}$$

Not being exactly the same form as above, the higher F-terms also appear in the low-energy limit of heterotic string theory [3]. The Gauss-Bonnet black holes with the higher F-terms have been investigated in [11]. Our action (1.3) may be considered as the simplest model of such higher F-terms.

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The nonlinear source (1.3) has been considered in general relativity [12] where it has been derived black-hole solutions with interesting asymptotic behaviors. In general, black hole solutions with nonlinear electrodynamics sources have been extensively analyzed in the current literature, see e.g. [13] and references therein. Nonlinear electrodynamics sources are also good laboratories in order to construct black-hole solutions with appealing features as for instance regular black holes [14]. Moreover, the nonlinear electrodynamics models exhibit interesting thermodynamics properties since they satisfy both the zeroth and first laws of black-hole mechanics [15].

The plan of the paper is organized as follows. In the next section, we consider the Einstein-Gauss-Bonnet action with the nonlinear electrodynamics source (1.3). In this case, we derive electrically charged black-hole solutions and a generalized version of the Birkhoff's theorem is proved. In the section III, the properties of the solution are discussed. In the section IV, our analysis is extended to the full Lovelock action where it is shown that the metric is given as a solution of a polynomial equation. In general, this polynomial equation may have no real roots, in which case the metric solution being purely imaginary is not physically admissible. Interesting enough, we show that this polynomial equation always admits at least one real root for the special election of the Lovelock coefficients given by (1.2). The summary and the future prospect of the present paper are given in section V.

II. GAUSS-BONNET BLACK HOLES WITH A NONLINEAR ELECTRODYNAMICS SOURCE

In this section, we consider the Einstein-Gauss-Bonnet equations with the nonlinear electrodynamics source (1.3) in arbitrary dimensions. The n-dimensional action is given by

$$S[g_{\mu\nu}, A_\mu] = \int d^n x\sqrt{-g}\left[\frac{1}{2\kappa_n^2}(R-2\Lambda+\alpha L_{GB})\right] - \beta\int d^n x\sqrt{-g}(F_{\mu\nu}F^{\mu\nu})^q, \tag{2.1}$$

where R and Λ are n-dimensional Ricci scalar and the cosmological constant, respectively. $F_{\mu\nu}$ is the strength of the nonlinear electromagnetic field and q is an arbitrary rational number whose range will be fixed later. Further $\kappa_n \equiv \sqrt{8\pi G_n}$, where G_n is n-dimensional gravitational constant and α and β are the coupling constants for the Gauss-Bonnet term L_{GB} and the nonlinear electromagnetic field, respectively. The Gauss-Bonnet term L_{GB} is combination of squares of Ricci scalar, Ricci tensor $R_{\mu\nu}$, and Riemann tensor $R^\mu{}_{\nu\rho\sigma}$ as

$$L_{GB} := R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \tag{2.2}$$

The basic equations following from the action (2.1) are given by

$$\mathcal{G}^\mu{}_\nu := G^\mu{}_\nu + \alpha H^\mu{}_\nu + \Lambda\delta^\mu{}_\nu = \kappa_n^2 T^\mu{}_\nu, \tag{2.3}$$

$$0 = \frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}F^{\mu\nu}\mathcal{F}^{q-1}), \tag{2.4}$$

where for convenience we have defined $\mathcal{F} := F_{\mu\nu}F^{\mu\nu}$, and where the geometric quantities and the energy-momentum tensor of the nonlinear electromagnetic field are defined by

$$\begin{aligned} G_{\mu\nu} &:= R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, & (2.5)\\ H_{\mu\nu} &:= 2\Big[RR_{\mu\nu} - 2R_{\mu\alpha}R^\alpha_{\ \nu} - 2R^{\alpha\beta}R_{\mu\alpha\nu\beta} \\ &\qquad + R_\mu{}^{\alpha\beta\gamma}R_{\nu\alpha\beta\gamma}\Big] - \frac{1}{2}g_{\mu\nu}L_{GB}, & (2.6)\\ T_{\mu\nu} &:= 4\beta\left(qF_{\mu\rho}F_\nu{}^\rho\mathcal{F}^{q-1} - \frac{1}{4}g_{\mu\nu}\mathcal{F}^q\right). & (2.7)\end{aligned}$$

Now we consider an Ansatz for the spacetime geometry such that the $n(\geq 5)$-dimensional spacetime $(\mathcal{M}^n, g_{\mu\nu})$ is given as a warped product of an $(n-2)$-dimensional constant curvature space (K^{n-2}, γ_{ij}) and a two-dimensional orbit spacetime (M^2, g_{ab}) under the isometries of (K^{n-2}, γ_{ij}). Namely, the line element is given by

$$g_{\mu\nu}dx^\mu dx^\nu = g_{ab}(y)dy^a dy^b + \mathcal{R}^2(y)\gamma_{ij}(z)dz^i dz^j, \tag{2.8}$$

where $a, b = 0, 1;\ i, j = 2, ..., n-1$. Here $\mathcal{R}$ is a scalar on (M^2, g_{ab}) with $\mathcal{R} = 0$ defining its boundary and γ_{ij} is the unit metric on (K^{n-2}, γ_{ij}) with its sectional curvature $k = \pm 1, 0$.

In what follows, we first derive an electrically charged solution with a particular Ansatz of the form

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2\gamma_{ij}dz^i dz^j, \tag{2.9}$$

and then we prove that the solution obtained is the unique under the assumption that $D_a\mathcal{R}$ is not null. Here D_a stands for a metric compatible linear connection on the manifold (M^2, g_{ab}).

A. Electrically charged black-hole solutions

Here we only consider the electrically charged case, i.e., $F_{ij} \equiv 0$, and hence the non-zero components of the energy-momentum tensor are given by $T^a{}_b = \beta(2q-1)\mathcal{F}^q\delta^a{}_b$ and $T^i{}_j = -\beta\mathcal{F}^q\delta^i{}_j$. In this setting, we obtain the following solution for the Ansatz (2.9)

$$f(r) = k + \frac{r^2}{2\tilde{\alpha}}\left(1 \mp \sqrt{1 + 4\tilde{\alpha}\tilde{\Lambda} + \frac{\tilde{\alpha}M}{r^{n-1}} + \frac{\tilde{\alpha}B}{r^\gamma}}\right), \tag{2.10}$$

$$F_{tr} = \frac{C}{r^{(n-2)/(2q-1)}}, \tag{2.11}$$

for $q \neq 1/2$, where C is a constant and where we have defined

$$B := \frac{8\kappa_n^2 \beta C^{2q}(-2)^q(2q-1)^2}{(n-2)(n-1-2q)},\ \gamma := \frac{2q(n-2)}{2q-1}. \quad (2.12)$$

The remaining constants appearing in the solution are $\tilde{\Lambda} := 2\Lambda/[(n-1)(n-2)]$, $\tilde{\alpha} := (n-3)(n-4)\alpha$, while M stands for an arbitrary constant.

Various comments can be made concerning the solution obtained. Firstly, this solution reduces to the solutions obtained by Boulware and Deser, and independently by Wheeler for $C = 0$, $k = 1$, and $\Lambda = 0$ [16], by Wiltshire for $q = 1$, $k = 1$, and $\Lambda = 0$ [17], by Lorenz-Petzold and independently by Cai for $C = 0$ [18, 19] (the left-hand side of Eq. (10) in [18] should be u^{-2}), and by Cvetic, Nojiri, and Odintsov for $q = 1$ [20]. Subsequently, it is important to stress that since the only non-vanishing components of the Maxwell tensor is given by F_{tr}, the Maxwell invariant $\mathcal{F} = -2(F_{tr})^2$ is negative, and hence in order to deal with real solutions, the exponent q must be restricted to be an integer or a rational number with odd denominator. As a consequence, the singular case of $q = 1/2$ is excluded from the discussion.

It is also clear from the expression of the constant B (2.12) that the solution given by (2.10) is valid only for $n \neq 2q + 1$. For this particular case, which corresponds to an exponent $q \in \mathbb{N}$ in odd dimensions $n = 2q + 1$ (so $q \geq 2$), the solution reads

$$f(r) = k + \frac{r^2}{2\bar{\alpha}}\left(1 \mp \sqrt{1 + 4\bar{\alpha}\bar{\Lambda} + \frac{\bar{\alpha}M}{r^{2q}} - \frac{\bar{\alpha}\bar{B}\ln(r)}{r^{2q}}}\right), \quad (2.13)$$

where $\bar{\alpha} := 2(q-1)(2q-3)\alpha$, $\bar{\Lambda} := \Lambda/[q(2q-1)]$, and $\bar{B} := 8\kappa_n^2\beta C^{2q}(-2)^q$.

B. Uniqueness

We now show that the particular solution represented by (2.10)–(2.13) is the unique solution (up to isometries) under the assumption that D_aR is not null. In what follows, we only consider the case for which D_aR is spacelike since the derivation in the timelike case is quite analogue. In the neutral case, i.e., $C = 0$, this generalized Birkhoff's theorem was shown under the same assumption, i.e., $(D_aR)(D^aR) \neq 0$ in [17, 21, 22], while the complete proof including the null case was given in [23].

In the case where D_aR is spacelike, we can set R to be the radial space coordinate, and in this case the general metric reads

$$ds^2 = -N(t,r)g(t,r)dt^2 + \frac{1}{g(t,r)}dr^2 + r^2\gamma_{ij}dz^idz^j. \quad (2.14)$$

The contravariant-covariant component (t,r) or (r,t) of Eq. (2.3) imply that the metric function g does not depend on t, i.e. $g(t,r) = g(r)$. Subsequently, the combination $(\mathcal{G}^t{}_t - \mathcal{G}^r{}_r) - \kappa_n^2(T^t{}_t - T^r{}_r) = 0$ gives rise to two possibilities, $N(t,r) = N(t)$ or

$$g(r) = k + \frac{r^2}{2\tilde{\alpha}}. \quad (2.15)$$

In both cases, the nontrivial basic equations (2.3) are given by

$$\mathcal{G}^a{}_b = \kappa_n^2\beta(2q-1)\mathcal{F}^q\delta^a{}_b, \quad (2.16)$$
$$\mathcal{G}^i{}_j = -\kappa_n^2\beta\mathcal{F}^q\delta^i{}_j, \quad (2.17)$$
$$\mathcal{F} = -\frac{2}{N}(F_{tr})^2. \quad (2.18)$$

In the first case, namely $N(t,r) = N(t)$, we can set $N(t) \equiv 1$ without loss of generality. Then, Eq. (2.4) gives

$$0 = \partial_r(r^{n-2}F_{tr}(-2F_{tr}^2)^{q-1}), \quad (2.19)$$
$$0 = \partial_t(r^{n-2}F_{tr}(-2F_{tr}^2)^{q-1}), \quad (2.20)$$

from which we deduce that the Maxwell field strength is given by (2.11). Finally, the remaining metric function $g(r)$ is given by $g(r) = f(r)$, where $f(r)$ is expressed as Eq. (2.10) and (2.13) for $n \neq 2q + 1$ and $n = 2q + 1$, respectively.

We now analyze the remaining option $g(r) = k + r^2/(2\tilde{\alpha})$. In this case, considering the equation $\mathcal{G}^t{}_t + (2q-1)\mathcal{G}^2{}_2 = 0$ given from Eqs. (2.16) and (2.17), we obtain that $\tilde{\Lambda} = -1/(4\tilde{\alpha})$. Through the basic equations (2.16) and (2.17) with $g(r) = k + r^2/(2\tilde{\alpha})$ and $\tilde{\Lambda} = -1/(4\tilde{\alpha})$ imply that the system is vacuum, i.e., $T^\mu{}_\nu \equiv 0$, and $N(t,r)$ is arbitrary. This exceptional vacuum (non-static) solution under the special combination between the coupling constants α and Λ was first found in [21].

Here we have shown the uniqueness of our solution (2.10)–(2.13) under the assumption that D_aR is not null. For the null case, on the other hand, there must be the Nariai-Bertotti-Robinson type solution [24] as in the case with or without the Maxwell field in general relativity [25] and in the Einstein-Gauss-Bonnet gravity [26, 27].

C. Energy conditions

Before analyzing the properties of the solutions obtained in (2.10) and (2.13), we discuss the energy conditions for the nonlinear electromagnetic field. For the energy momentum tensor written in the diagonal form as $T^\mu{}_\nu = \mathrm{diag}(-\mu, p_\mathrm{r}, p_\mathrm{t}, p_\mathrm{t}, \cdots)$, the weak energy condition (WEC) implies $\mu \geq 0$, $p_\mathrm{r} + \mu \geq 0$, and $p_\mathrm{t} + \mu \geq 0$, while the dominant energy condition (DEC) implies $\mu \geq 0$, $-\mu \leq p_\mathrm{r} \leq \mu$, and $-\mu \leq p_\mathrm{t} \leq \mu$ [28]. The physical interpretations of μ, p_r, and p_t are energy density, radial pressure, and the tangential pressure, respectively. The WEC assures that a timelike observer measures the non-negative energy density. The DEC assures in addition that the energy flux is a future-directed causal vector. The DEC implies the WEC, but the converse is not true.

In our case, the corresponding μ, $p_{\rm r}$, and $p_{\rm t}$ are respectively given by

$$\mu = -\beta(2q-1)\mathcal{F}^q, \qquad (2.21)$$
$$p_{\rm r} = \beta(2q-1)\mathcal{F}^q, \qquad (2.22)$$
$$p_{\rm t} = -\beta\mathcal{F}^q, \qquad (2.23)$$
$$\mathcal{F} = -2(F_{tr})^2. \qquad (2.24)$$

It is noted again that the exponent q must be restricted to be an integer or a rational number with odd denominator in order to deal with real solutions, so that $q = 1/2$ is excluded.

First we consider the condition $\mu \geq 0$, which determines the sign of β depending on the range of the exponent q,

$$\begin{cases} \mathrm{sgn}(\beta) = -(-1)^q & \text{for} \quad q > 1/2, \\ \mathrm{sgn}(\beta) = (-1)^q & \text{for} \quad q < 1/2. \end{cases} \qquad (2.25)$$

Then, both WEC and DEC are satisfied for $q \leq 0$ or $q \geq 1$. The WEC is satisfied but the DEC is violated for $1/2 < q < 1$. On the other hand, both WEC and DEC are violated for $0 < q < 1/2$.

We also consider the strong energy condition (SEC) which implies $p_{\rm r} + \mu \geq 0$, $p_{\rm t} + \mu \geq 0$, and $\mu + p_{\rm r} + (n-2)p_{\rm t} \geq 0$. It is noted that the SEC is independent from either WEC or DEC. Independent of the sign of β, the SEC is satisfied for $q > 1/2$ with $\mu > 0$ or $0 \leq q < 1/2$ with $\mu < 0$, otherwise it is violated. The result obtained in this subsection is summarized in table I.

TABLE I: Consistency for the nonlinear electromagnetic field with the energy conditions under the condition (2.25) corresponding to $\mu > 0$. For the real solutions, the exponent q must be an integer or a rational number with odd denominator. We note that the strong energy condition is satisfied even for $0 \leq q < 1/2$ if $\mu < 0$ holds.

	$q \leq 0$	$0 < q < 1/2$	$1/2 < q < 1$	$1 \leq q$
WEC	Yes	No	Yes	Yes
DEC	Yes	No	No	Yes
SEC	No	No	Yes	Yes

To conclude the study of the energy condition, we would like to stress that the excluded region $0 < q < 1/2$ where none of the energies conditions are satisfied is also ruled out by the following argument. Since we are interested in finding solutions with event horizons that should hide the eventual singularities, solutions having singularities at infinity will be ruled out and only curvature singularities surrounded by an event horizon will be allowed. For $q \in]0, 1/2[$, the scalar curvature associated to the solution (2.10) diverges at infinity or the metric may be complex at infinity depending on the parameters.

III. PROPERTIES OF THE SOLUTION

In this section, we analyze the solutions obtained (2.10) and (2.13). There are two families of solutions corresponding to the sign in front of the square root in Eq. (2.10) or (2.13), stemming from the quadratic curvature terms in the action. The solution with the upper sign, that we call the GR branch, has a general relativistic (GR) limit as $\alpha \to 0$ given by

$$f(r) = k - \tilde{\Lambda}r^2 - \frac{M}{4r^{n-3}} - \frac{B}{4r^{\gamma-2}}, \qquad (3.1)$$
$$f(r) = k - \bar{\Lambda}r^2 - \frac{M}{4r^{2q-2}} + \frac{\bar{B}\ln(r)}{r^{2q-2}} \qquad (3.2)$$

for $n \neq 2q+1$ and for $n = 2q+1$ respectively. This is a generalization of the solution obtained in [12] for $k = 1$ and $\Lambda = 0$. In contrary, the other branch, i.e. the lower signs in (2.10) and (2.13), that we call the Gauss-Bonnet branch, does not have the GR limit.

Setting $\tilde{\Lambda} = \bar{\Lambda} = M = C = 0$ in (2.10) or (2.13), the possible vacua differ drastically from one case to the other. Indeed, in the GR branch, the metric will reduce to that of Minkowski while for the Gauss-Bonnet branch, the metric becomes that of (A)dS with an effective cosmological constant that goes like $-(1/\alpha)$. Indeed, in this case a small coupling constant α will correspond to a huge effective cosmological constant.

We now turn to the crucial question about the singularities and the existence of event horizons. In order to achieve this task correctly and because of the presence of many parameters in the metric solution (2.10), we put several conditions on the parameters. First we assume $1 + 4\tilde{\alpha}\tilde{\Lambda} \geq 0$ and $1 + 4\bar{\alpha}\bar{\Lambda} \geq 0$ which ensure the existence of the maximally symmetric solutions. We also assume the weak energy condition for the nonlinear electromagnetic field, under which $q \leq 0$ or $q > 1/2$ is satisfied and γ is non-negative.

Under the reasonable assumptions listed above, the parameter space of the solution may be classified into several cases depending on the fall-off rate of the electromagnetic term against the gravitational term. The first case corresponding to $\gamma > n-1$ is similar to the standard Maxwell case and will be achieved for the exponent $q \in]1/2, (n-1)/2[$. On the other hand, the option $\gamma < n-1$ can also be considered with $q \leq 0$ or $q > (n-1)/2$ while the case $\gamma = (n-1)$ will correspond to the logarithmic metric (2.13).

In a generic way, the solution may have two possible singularities that are the usual $r = 0$ and also a branch singularity at $r = r_{\rm b}(> 0)$, where the argument of the square-root piece of the metric solution (2.10) or (2.13) vanishes. For $r < r_{\rm b}$, the metric becomes complex.

In order to clarify the existence condition for the branch singularity, we write the function $f(r)$ as

$$f(r) \;=\; k+\frac{r^2}{2\tilde{\alpha}}\left(1\mp\sqrt{1+4\tilde{\alpha}\tilde{\Lambda}+\frac{\tilde{\alpha}M}{r^{n-1}}-\frac{8\tilde{\alpha}\kappa_n^2(2q-1)\mu}{(n-1)(n-1-2q)}}\right), \tag{3.3}$$

$$f(r) \;=\; k+\frac{r^2}{2\bar{\alpha}}\left(1\mp\sqrt{1+4\bar{\alpha}\bar{\Lambda}+\frac{\bar{\alpha}M}{r^{2q}}+\frac{8\kappa_n^2\bar{\alpha}\mu\ln(r)}{2q-1}}\right) \tag{3.4}$$

for $n\neq 2q+1$ and for $n=2q+1$ respectively, where $\mu(r)$ is the energy density of the nonlinear electromagnetic field (2.21).

For $q\in]1/2,(n-1)/2[$ corresponding to $\gamma>n-1$, the electromagnetic term dominates inside the square-root for $r\to 0$. As a result, the branch singularity exists for $\tilde{\alpha}>0$. On the other hand, for $q\le 0$ or $q>(n-1)/2$ corresponding to $\gamma<n-1$, the gravitational term dominates for $r\to 0$ and the branch singularity exists for $\tilde{\alpha}M<0$. Finally, in the case of $q=(n-1)/2$ corresponding to the logarithmic metric (3.4), the electromagnetic term dominates for $r\to 0$, so that the branch singularity exists for $\bar{\alpha}>0$ since $2q-1>0$ is satisfied.

As a consequence of the branch singularity, the event horizon given by the positive real root of the algebraic equation $f(r_{\rm h})=0$ must satisfy an inequality $r_{\rm h}>\max(0,r_{\rm b})$. The location of horizon $r_{\rm h}$ is a root of the following polynomial

$$p(r):=4k^2\tilde{\alpha}r^{\gamma-4}+4kr^{\gamma-2}-4\tilde{\Lambda}r^{\gamma}-Mr^{\gamma+1-n}-B=0 \tag{3.5}$$

for $\gamma>n-1$, while for $\gamma<n-1$, the polynomial reads

$$\begin{aligned}p(r):=&\,4k^2\tilde{\alpha}r^{n-5}+4kr^{n-3}\\&-4\tilde{\Lambda}r^{n-1}-Br^{n-\gamma-1}-M=0.\end{aligned} \tag{3.6}$$

For the logarithmic case, i.e., $\gamma=n-1$, the horizon $r_{\rm h}$ is the solution of

$$4k^2\bar{\alpha}r^{2q-4}+4kr^{2q-2}-4\bar{\Lambda}r^{2q}+\bar{B}\ln(r)-M=0, \tag{3.7}$$

which is not a polynomial. Moreover, in all the cases, the roots must satisfy the condition $\mp[k+r_{\rm h}^2/(2\tilde{\alpha})]\le 0$, where the upper and the lower signs in the left-hand side correspond to the GR and the Gauss-Bonnet branches, respectively. This extra condition ensures the equivalence between the roots of the polynomial $p(r)$ and those of the metric function $f(r)$. A full analysis of the existence condition for the event horizon will certainly be interesting but it is rendered long by the presence of so many parameters. Hence, in order to gain in clarity we avoid a more detailed discussion.

In the case of $q=1$, the global structure of the solution was fully investigated in [29]. (See [30] for the neutral case.) In fact, the number of the horizons, structure of the singularity, and asymptotic behavior at infinity, sharply depend on the parameters in the solution.

IV. EXTENSION TO LOVELOCK GRAVITY

The extension of the analysis in the previous section to the more general Lovelock gravity is an interesting subject by itself. In this objective, we shall consider the Lovelock gravity (1.1) with the nonlinear electrodynamics source (1.3) in arbitrary dimensions and look for particular solutions. The Ansatz for the geometry we shall consider is the same that in the Gauss-Bonnet case (2.9), and we shall also restrict the nonlinear electromagnetic field to be a purely radial one. As in the Gauss-Bonnet case, this Ansatz will restrict the exponent q to be given as a rational number with odd denominator.

In this analysis, we opt for the Hamiltonian formalism that provides an easy way to write down the field equations and integrate them. In order to achieve this task, we first write the reduced Lovelock Hamiltonian [31],

$$\mathcal{H}^L=-(n-2)!\sqrt{\frac{\gamma}{f}}\frac{d}{dr}\left[r^{n-1}\sum_{p=0}^{[\frac{n-1}{2}]}\alpha_p(n-2p)\left(\frac{k-f}{r^2}\right)^p\right], \tag{4.1}$$

where γ is the determinant of γ_{ij}. In an analogue way, the reduced nonlinear electromagnetic Hamiltonian is given by

$$\mathcal{H}^e=-\beta\sqrt{\frac{\gamma}{f}}\frac{(2q-1)\,(-2)^{\frac{q}{2q-1}}\,\mathcal{P}^{\frac{2q}{2q-1}}}{(4q\beta)^{\frac{2q}{2q-1}}\,r^{\frac{n-2}{2q-1}}}, \tag{4.2}$$

where $\mathcal{P}:=4\beta q\mathcal{F}^{q-1}r^{n-2}F_{tr}$ is the rescaled radial momentum which is constant by virtue of the Gauss law. Defining a function $H(r)$ such that $f(r)=k-r^2H(r)$, the constraint becomes a first-order equation given by

$$\begin{aligned}&\frac{d}{dr}\left[r^{n-1}\sum_{p=0}^{[\frac{n-1}{2}]}\alpha_p(n-2p)H^p\right]\\&\qquad=\frac{\beta}{(n-2)!}\frac{(2q-1)\,(-2)^{\frac{q}{2q-1}}\,\mathcal{P}^{\frac{2q}{2q-1}}}{(4q\beta)^{\frac{2q}{2q-1}}\,r^{\frac{n-2}{2q-1}}},\end{aligned} \tag{4.3}$$

whose straightforward integration yields

$$\sum_{p=0}^{[\frac{n-1}{2}]}\alpha_p(n-2p)H^p=\frac{C_1}{r^{n-1}}+\frac{\beta(2q-1)^2\,(-2)^{\frac{q}{2q-1}}\,\mathcal{P}^{\frac{2q}{2q-1}}}{(n-2)!(4q\beta)^{\frac{2q}{2q-1}}\,(n-1-2q)\,r^{\frac{2q(n-2)}{2q-1}}},\qquad \text{for}\quad n\neq 2q+1 \tag{4.4}$$

$$\sum_{p=0}^{[\frac{n-1}{2}]}\alpha_p(n-2p)H^p=\frac{C_1}{r^{n-1}}-\frac{\beta(2q-1)(-2)^{\frac{q}{2q-1}}\,\mathcal{P}^{\frac{q}{2q-1}}\,\ln r}{(2q-1)!\,(4q\beta)^{\frac{q}{2q-1}}\,r^{2q}},\qquad \text{for}\quad n=2q+1, \tag{4.5}$$

where C_1 is an integration constant in both cases. In both cases, the electric field is given by the same expression as in the Gauss-Bonnet case (2.11).

A. Dimensionally continued gravity

In principle, one may find up to $[\frac{n-1}{2}]$ real roots and in dimensions $n = 4m+3$ and $4m+4$ with an integer $m(\geq 1)$, these equations will always admit at least one real root. However, it is interesting to observe that an enormous simplification occurs in these equations as the Lovelock coefficients α_p takes the particular values (1.2) that convert the Lovelock action into a Chern-Simons gauge theory in odd dimensions. Indeed, in this case, and for odd as well as even dimensions, the left-hand sides of the equations (4.4) and (4.5) become the Newton binomial expression,

$$\sum_{p=0}^{[\frac{n-1}{2}]}\alpha_p(n-2p)H^p\equiv\Big(1+H\Big)^{[\frac{n-1}{2}]}. \tag{4.6}$$

Consequently, in both cases, the function H can be determined explicitly. The metric solution f can be written in odd dimension by

$$f(r)=k+r^2-\Big(C_1+\frac{C_2}{r^{\frac{n-1-2q}{2q-1}}}\Big)^{\frac{2}{n-1}}, \tag{4.7}$$

while the expression in even dimension is giving by

$$f(r)=k+r^2-\frac{1}{r^{\frac{2}{n-2}}}\Big(C_1+\frac{C_2}{r^{\frac{n-1-2q}{2q-1}}}\Big)^{\frac{2}{n-2}}, \tag{4.8}$$

where C_2 stands in both cases for

$$C_2:=\frac{\beta(2q-1)^2\,(-2)^{\frac{q}{2q-1}}\,\mathcal{P}^{\frac{2q}{2q-1}}}{(n-2)!(4q\beta)^{\frac{2q}{2q-1}}\,(n-1-2q)}. \tag{4.9}$$

This solution is a generalization of the solution obtained by Bañados, Teitelboim, and Zanelli for $k=1$ and $q=1$ in [31] and by Cai and Soh for $q=1$ in [32]. In odd dimensions, this solution is the higher-dimensional counterpart with a nonlinear electromagnetic charge of the so-called BTZ black hole in three dimensions [33]. It is very appealing that for the special election of the Lovelock coefficients, the metric function can be integrated easily in odd as well as even dimensions.

B. Properties of the solution

Now let us discuss the properties of the solution (4.7) and (4.8) under the weak energy condition. The constant C_2 can be written in terms of the energy density of the nonlinear electromagnetic field (2.21) as

$$C_2=-\frac{(2q-1)\mu r^{\frac{2q(n-2)}{2q-1}}}{(n-2)!(n-1-2q)}. \tag{4.10}$$

Therefore, under the weak energy condition, $C_2<0$ for $q\in]1/2,(n-1)/2[$ while $C_2>0$ for $q\leq 0$ or $q>(n-1)/2$.

In the case for which the dimension is expressed as $n=4m+1$ or $n=4m+2$, where $m(\geq 1)$ is an integer, the exponent in the binomial expression (4.6) is even and hence the equations (4.5) have two branches of solutions

$$f(r)=k+r^2\mp\Big(C_1+\frac{C_2}{r^{\frac{4m-2q}{2q-1}}}\Big)^{\frac{1}{2m}},\quad \text{for}\quad n=4m+1, \tag{4.11}$$

$$f(r)=k+r^2\mp\Big(\frac{C_1}{r}+\frac{C_2}{r^{\frac{4m}{2q-1}}}\Big)^{\frac{1}{2m}},\quad \text{for}\quad n=4m+2. \tag{4.12}$$

Note that the Einstein-Gauss-Bonnet solution derived previously (2.10) in five dimensions with the special election $\tilde\alpha=-1/(4\tilde\Lambda)$ reduces to the first expression (4.11) with $m=1$. This is not surprising since the condition $\tilde\alpha=-1/(4\tilde\Lambda)$ is nothing but the Chern-Simons limit of the Gauss-Bonnet theory in five dimensions.

For the solutions (4.11-4.12), we have to care about the possible branch singularities. In these cases, the gravitational term dominates the nonlinear electromagnetic term at infinity for $q\in]1/2,(n-1)/2[$, so that C_1 must be positive in order that the metric is real at infinity. On the other hand, there exists a branch singularity since $C_2<0$ is required by the weak energy condition. For $q\leq 0$ or $q>(n-1)/2$, the nonlinear electromagnetic term dominates the gravitational term at infinity and the metric is real because the weak energy condition requires $C_2>0$. In this case, there exists a branch singularity for $C_1<0$ and a central singularity for $C_1>0$.

In the case with $n=4m+3$ or $n=4m+4$, on the other hand, both C_1 and C_2 may be negative since we may rewrite (4.7) and (4.8) as

$$f(r)=k+r^2+\Big(-C_1-\frac{C_2}{r^{\frac{n-1-2q}{2q-1}}}\Big)^{\frac{2}{n-1}}, \tag{4.13}$$

and

$$f(r) = k + r^2 + \frac{1}{r^{\frac{2}{n-2}}}\Big(-C_1 - \frac{C_2}{r^{\frac{n-1-2q}{2q-1}}}\Big)^{\frac{2}{n-2}}, \quad (4.14)$$

respectively. The branch singularity exists only for $C_1 C_2 < 0$. Unlike the case of $n = 4m+1$ or $n = 4m+2$, there is no region where the metric becomes complex even if there exists a branch singularity. Under the weak energy condition, the branch singularity exists for $q \in]1/2, (n-1)/2[$ with $C_1 > 0$ and for $q \leq 0$ or $q > (n-1)/2$ with $C_1 < 0$.

In the solution (4.7) and (4.8), the fall-off rate to infinity is slower than the standard one. Even under the weak energy condition, the electromagnetic term diverges at infinity for $q < 1/2$ or $q > (n-1)/2$. However, it is shown that the divergence is faster than r^2 only for $0 < q < 1/2$ both in odd and even dimensions, which is ruled out by the weak energy condition. As a result, the regular infinity is assured by the weak energy condition. This slow fall-off phenomenon was first pointed out in the study of the static black holes with and without the Maxwell field in the class of Lovelock gravity admitting a unique (A)dS vacuum [34]. Recently, this phenomenon was shown to be universal for any matter field satisfying the dominant energy condition in Einstein-Gauss-Bonnet gravity with $1 + 4\tilde{\alpha}\tilde{\Lambda} = 0$ and $\alpha > 0$ [35].

We finally end this section with some speculation concerning a possible Birkhoff's theorem. In Lovelock gravity, the generalized Birkhoff's theorem has been proven under the same assumption of the present paper in the vacuum case and for the standard Maxwell case [36, 37]. Because of the similarity in the treatment, we may envisage that our charged solution is the unique electrically charged solution within the nonlinear source considered here.

V. SUMMARY AND FURTHER PROSPECTS

In the present paper, we obtained electrically charged black-hole solutions in Einstein-Gauss-Bonnet gravity with a nonlinear source given as an arbitrary exponent q of the Maxwell invariant. We have considered the class of the $n(\geq 5)$-dimensional spacetime given as a warped product $\mathcal{M}^2 \times \mathcal{K}^{n-2}$. The generic solution is shown to have two branches and only one of them has a GR limit. For an integer value of q with a dimension $n = 2q+1$, the metric solution involves a logarithmic dependence and as in the generic case, the solution presents as well two different branches. We show that these solutions are the unique electrically charged solution in the case where the orbit of the warp factor on $\mathcal{K}^{n-2}$ is non-null.

We find that an intriguing slow fall-off to the spacelike infinity is possible even under the dominant energy condition. This slow fall-off was shown to be universal for any matter field satisfying the dominant energy condition in the special case of $1 + 4\tilde{\alpha}\tilde{\Lambda} = 0$ and $\alpha > 0$, which corresponds to the Chern-Simons gravity in five dimensions [34, 35]. Our solution is an example exhibiting the slow fall-off with generic coupling constant. We have analyzed the properties of the solutions emphasizing the study on the branch singularities.

We have also derived charged black hole solutions for the full Lovelock gravity. In this case, the metric function is obtained implicitly as a solution of a polynomial equation. We have pointed out that for a very precise combination between the coupling constants, which converts the Lovelock action into a Chern-Simons gauge theory in odd dimensions, the metric function is obtained in a closed form both in odd and even dimensions. It has been shown that a branch singularity appears under the weak energy condition depending on the parameters. Unlike the Gauss-Bonnet case, the metric does not become complex for $n = 4m+3$ and $n = 4m+4$ with an integer $m(\geq 1)$ even if there is a branch singularity. The slow fall-off to the regular infinity is a generic property under the weak energy condition both in odd and even dimensions.

This slow fall-off has recently attracted much attention [38] for theories of AdS gravity coupled to a scalar field with mass at or slightly above the Breitenlohner-Freedman bound [39]. These theories admit a large class of asymptotically AdS spacetimes with slower fall-off conditions than the standard ones.

The dynamical stability of the black-hole solution is an important problem. In the standard Einstein-Maxwell case, the stability analysis has been carried out in four dimensions [40] as well as in higher dimensions [41], see Table I in [42] for the summary of the analytic results. In the Einstein-Gauss-Bonnet gravity, the analysis has been done only in the neutral case [43], while there is no result for higher-order Lovelock gravity. In the case of the nonlinear electromagnetic field, the stability analysis has not been done even in four dimensions. In this context, the asymptotic slow fall-off would be important because it could affect the boundary conditions for the perturbations.

The black-hole thermodynamics is another interesting subject. In Einstein-Gauss-Bonnet gravity, this subject have been intensively investigated with or without the Maxwell charge [19, 20, 30, 44]. The extension to the full Lovelock gravity is also well studied [31, 32, 45]. In this context, the slow fall-off to the spacelike infinity becomes important. Under the standard fall-off condition, the higher-dimensional ADM mass is available as the global mass in the asymptotically flat case [46]. In the asymptotically (A)dS spacetime, several definitions of the global mass have been proposed in Einstein-Gauss-Bonnet gravity [47, 48, 49]. However, the slow fall-off means that they are diverging at infinity. In order to discuss the thermodynamical properties of black holes correctly, one should first reformulate the global mass in order to give a finite value under the slower fall-off condition in this special case. This problem has been investigated in Chern-Simons gravity [50] and in the theory admitting

a unique (A)dS vacuum [34]. (See also [51, 52].)

Other aspects to explore are the extensions of the solutions presented here in more general context. For example, it will be interesting to explore the possible dilatonic solutions in this set-up or the existence of magnetically charged solutions. These prospects presented here are left for possible future investigations.

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